Synchronization-based noise reduction method for communication with chaotic systems

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We describe a noise reduction algorithm for communication using a controlled chaotic system. Our algorithm uses the phenomenon of chaos synchronization, and utilizes the knowledge of the transmitter dynamics in extracting the signal. The correct noise estimate at the receiver yields synchronization and has a minimum norm. A numerical experiment illustrating the method is presented, and shows that successful recovery of the transmitted signal is possible for signal to noise ratios of order unity. [S1063-651X(98)07911-2]

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Noise limits the information content that can be retrieved from a signal. In particular, this is an important consideration in the context of communication with chaotic signals [1,2]. There have been many methods proposed in which the dynamics, reconstructed from the observations or known *a priori*, can be used to identify and correct errors in noisy chaotic data [3]. A noise reduction method attempts to separate the received signal into noise and signal unambiguously, so that the signal becomes more consistent with the dynamical models. We describe a simple method based on chaos synchronization, restricting ourselves to the case of additive noise. The noise reduction method we present would be of particular use for the communication scheme proposed in Ref. [2]. In that communication scheme, the transmitter behaves chaotically, and a small control is used to cause the transmitter dynamics to follow one of the dynamically allowed symbol sequences of the symbolic dynamics of the free-running (i.e., uncontrolled) chaotic device. The transmitted information is thus encoded in the symbolic dynamic sequence available from the transmitted signal. Since the control is small, only dynamically allowed sequences are used, and the transmitted signal is essentially realizable as an orbit of the free-running system. (In effect, the role of the small control is to choose which of the infinite number possible free-running transmitter orbits is followed.) Thus, in what follows, we can assume that we wish to transmit some chaotic signal, and we make no further reference to the small control.

Let $x(t)$ be an output of the transmitter chaotic system. Let there be an additive noise $n(t)$, added in the channel through which $x(t)$ is transmitted. Hence the noisy time series available at the receiver is $x(t)+n(t)$. We assume here that $x(t)$ is scalar [the method we describe below can be extended to vector $x(t)$]. The problem which we address is the following: can we generate an estimate $\overline{n}(t)$, of the noise $n(t)$, at the receiver, so as to extract $x(t)$? As we describe below, the phenomenon of chaos synchronization [1,4] can be effectively exploited for generating such an estimate. Synchronization has been used before for the equalization of linear distortion in the channel [5] and for parameter recovery [6]. We extend this approach further for noise reduction.

From past work on chaos synchronization [1,4], we know that it is possible to design a synchronizing receiver. Thus, if the received signal is used to drive the synchronizing receiver, we know that, under no noise conditions, the input and the output of the synchronizing receiver will be the same. This will not be true, however, when noise is present. The synchronization error typically increases monotonically with input noise [7]. Consider now that an estimate $\overline{n}(t)$ of the noise $n(t)$ is continuously generated at the receiver, and let the final signal driving the synchronizing receiver be of the form $x(t)+n(t)-\overline{n}(t)$ (see Fig. 1). Let $\overline{n}(t)$ be chosen by some means such that synchronization is observed, i.e.,

$$y(t)=x(t)+n(t)-\overline{n}(t), \tag{1}$$

where $y(t)$ denotes the output of the synchronizing receiver. Since only those signals which satisfy the transmitter state equations pass undistorted through the synchronizing receiver, we conclude that, if Eq. (1) is satisfied, then $y(t)$ is a trajectory of the chaotic transmitter dynamical system. There will be, in general, a large number of signals $\overline{n}(t)$ for which the constraint (1) is satisfied. If $\overline{n}(t)=n(t)$, then $y(t)=x(t)$, and both $n(t)$ and $\overline{n}(t)$ have the same norm. Condition (1) holds for $\overline{n}(t)$ such that $y=x+n-\overline{n}$ is *any* possible orbit of the chaotic system, even one far from $x(t)$. If $y(t)$ and $x(t)$ are different orbits of the same chaotic system, then, due to exponential divergence, we expect that they will typically be very different. An exception is when $y(t)=x(t-\tau)$ and τ is small, in which case, $x(t)$ and $y(t)$ may remain close to each other for all time. We show that, in both these cases, $\overline{n}(t)$ has a larger norm than $n(t)$. [Another exception is where the orbits for $x(t)$ and $y(t)$ are on each others stable manifold, in which case $|x(t)-y(t)|\to 0$ with increasing t. Since we are interested in long time behavior this case is equivalent, for our purposes, to x and y being the

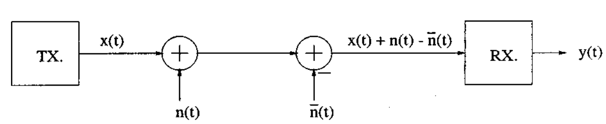

FIG. 1. The estimate of the noise is subtracted from the received signal.

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same.] We claim the following: Given constraint (1), then $\overline{n}(t)=n(t)$ only if $\|\overline{n}(t)\|$ is minimum. The norm of $\overline{n}(t)$ is given by

$$\|\overline{n}(t)\|^2=\frac{1}{T}\int_{t=0}^{T}(x(t)+n(t)-y(t))^2dt$$
$$=\sigma^2+\frac{1}{T}\int_{t=0}^{T}(x(t)-y(t))^2dt, \qquad (2)$$

where σ^2 is the variance of the zero mean noise, assumed to be uncorrelated with $x(t)$ and $y(t)$. The pathological case of $n(t)=y(t)-x(t)$, where $y(t)$ is some other trajectory of the transmitter chaotic system, is ruled out because of the assumption that $n(t)$ is uncorrelated with transmitter trajectories. In practice, in a physical communication channel like a telephone line, for example, the added thermal noise has no correlation with the trajectories of the transmitter system. The integral term in Eq. (2) is always positive if $x(t)\neq y(t)$. For $x(t)\neq y(t)$, the two chaotic trajectories $x(t)$ and $y(t)$ will be typically uncorrelated, and hence the integral term becomes $2(R_{xx}(0)-\mu_x^2)$ $[=2\,\mathrm{Var}(x)]$, where $R_{xx}(\tau)$ denotes $\langle x(t)x(t-\tau)\rangle$, the autocorrelation function of $x(t)$ and μ_x denotes $\langle x(t)\rangle$, the mean of $x(t)$. (The angular brackets denote the average over time.) However, it is possible that $y(t)=x(t-\tau)$, for some nonzero τ, since a time shifted version of $x(t)$ will also pass through the synchronizing receiver undistorted. The integral term, in this case, becomes equal to $2(R_{xx}(0)-R_{xx}(\tau))$. In each case, the integral term is a strictly positive quantity for a wrong estimate, which establishes the claim that $\overline{n}(t)$ is minimum only when it is the correct estimate. Hence our decoding algorithm should find the trajectory which is the nearest to the received noisy trajectory $x(t)+n(t)$. Note that the above argument is not dependent on the assumption of a specific form of noise $n(t)$ (Gaussian, uniform etc.) [8].

Since the number of $\overline{n}(t)$'s satisfying Eq. (1) will be very large, an exhaustive search to find the $\overline{n}(t)$ of minimum norm may not be feasible. We now suggest a simple (nonexhaustive) method to estimate the $\overline{n}(t)$ with a minimum norm which we find works well in numerical examples. Let the noisy time series of length N available at the receiver be $\{x(i)+n(i)\}_{i=1}^{N}$, where $x(i)$ may be a discrete time signal or an appropriately sampled version of a continuous time signal. The estimate of the noise $\overline{n}$ is generated by minimizing the function

$$T(\overline{n})=\sum_{i=1}^{N}(y(i)+\overline{n}(i)-x(i)-n(i))^2+\lambda\overline{n}^2(i),$$

where λ is a regularization parameter added to obtain the minimum norm solution. N will typically be a large number. Instead of minimizing over a large number of variables and hence increasing the complexity of the minimization algorithm, we do it over a smaller number of variables M, and then use a sliding window to obtain the complete estimate of the N variables. The function we use for minimizing M variables is

$$T_j(\overline{n}(j),\overline{n}(j+1),\ldots,\overline{n}(j+M-1))=\sum_{i=j-K}^{j-1}e^{\alpha(i-j)}(y(i)+\overline{n}(i)-x(i)-n(i))^2$$
$$+\sum_{i=j}^{j+M-1}[(y(i)+\overline{n}(i)-x(i)-n(i))^2+\lambda\overline{n}^2(i)]$$
$$+\sum_{i=j+M}^{j+M+K-1}e^{\alpha(j-i)}(y(i)+\overline{n}(i)-x(i)-n(i))^2. \qquad (3)$$

(Other choices are also possible.) We use an iterative procedure consisting of successive minimizing ''passes,'' where in each pass we minimize T_j, incrementing j from 1 to $N-M+1$, and we use the estimate of $\overline{n}(t)$ from the previous pass as the initial condition for minimization on the current pass. In the expression of T_j [Eq. (3)], the first and last summation terms are added since the information about a sample is also contained in the preceding and succeeding terms, and the exponential factor indicates the decay of this information. K limits the sum when the exponential term becomes small and varies $\sim 1/\alpha$. We use the previously generated estimates $\overline{n}(j-K),\ldots,\overline{n}(j-1)$ and $\overline{n}(j+M),\ldots,\overline{n}(j+M+K-1)$ for calculation and minimize over the variables $\overline{n}(j),\ldots,\overline{n}(j+M-1)$. The state vector of the synchronizing receiver at time instant $i=1$, denoted by $\mathbf{w}(1)$, is unknown, and is used to solve for the output of the synchronizing receiver at later instants. We estimate $\mathbf{w}(1)$ in each pass by minimizing the function

$$G(\mathbf{w}(1))=\sum_{i=1}^{N}(y(i)-x(i)-n(i))^2.$$

Our typical strategy involves choosing λ as initially large, and then gradually decreasing it after each pass by a scaling factor. For minimization, we use the downhill simplex method requiring only function computations. This method is discussed in standard references such as Ref. [9], and is better for our purposes than an algorithm like the steepest

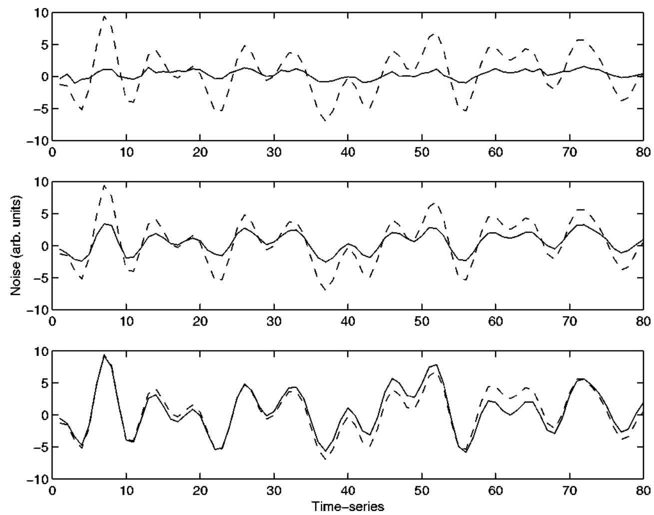

FIG. 2. The estimated noise (solid line) as the number of passes over the time series are increased: (a) 1, (b) 5, and (c) 15. The dashed line represents the additive Gaussian noise with $\sigma = 3.69$.

descent algorithm which requires the computation of derivatives and may be more time consuming.

As an illustration, the method was applied to the continuous time Lorenz system [10]. The Lorenz equations (variables X, Y, Z) and the equations for the synchronizing receiver [variables $X_r(t)$, $Y_r(t)$, $Z_r(t)$] are as follows (Cuomo and Oppenheim [1]):

$$\dot{X} = \sigma(Y - X), \quad \dot{Y} = rX - Y - XZ, \quad \dot{Z} = XY - bZ,$$

$$\dot{X}_r = \sigma(Y_r - X_r), \quad \dot{Y}_r = rX - Y_r - XZ_r, \quad \dot{Z}_r = XY_r - bZ_r,$$

where $(\sigma, r, b) = (16.0, 45.0, 4.0)$. The time series in this case consists of the sampled received signal. The sampling time was 0.01 (arbitrary units). The sampled signal was then interpolated, and used for integration to calculate the output of the synchronizing receiver. Polynomial interpolation of order two (quadratic) was used, where the point $X(n+\alpha)$ at a distance $\alpha \in [0,1]$ from $X(n)$ is determined as

$$X(n+\alpha) = 0.5\alpha(\alpha+1)X(n+1) + (1-\alpha^2)X(n) - 0.5\alpha(1+\alpha)X(n-1).$$

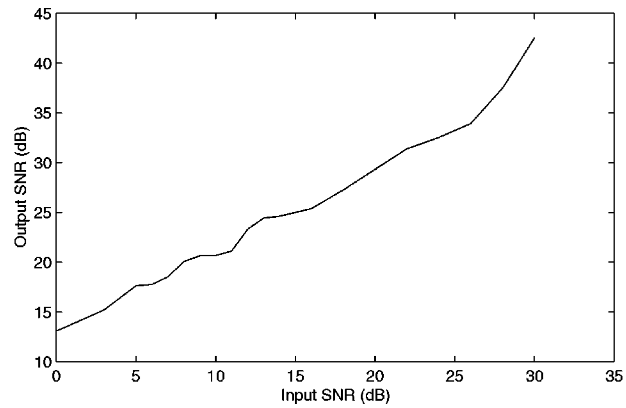

FIG. 3. The output SNR as a function of the input SNR.

We minimized the function T_j as defined by Eq. (3) when the signal to noise ratio (SNR) of the received signal was 10 dB [11]. The parameters α, K, M, and N were chosen as 0.25, 5, 6, and 80, respectively. The parameter λ was decreased by a factor of 2 after each pass with an initial value of 5.0. Figure 2 shows the actual and the estimated noise at various stages of the minimization. As shown in Fig. 2, the estimate of the noise $\overline{n}(t)$ approaches the actual noise $n(t)$ as the number of passes are increased. Initially the estimate resembles a scaled down version of the actual noise; as λ decreases, the estimate becomes better. The final value of the SNR after 15 passes was 20.7 dB. Figure 3 plots the input versus the output SNR. Typical gain in the SNR by this filtering operation is around 10 dB. The performance is limited by the interpolation inaccuracy, and a higher order interpolation would be expected to give a larger gain in the final SNR. The length N of the time series should not be too large. Since the state vector at the initial time is estimated at each pass, the interpolation error will be larger for large N, resulting in a worse estimate. In practice, this can always be taken care of by breaking a long time series into smaller lengths.

In conclusion, we have introduced and illustrated a simple method for reducing noise in chaotic signals. Our method is based on the minimum norm property satisfied by the constrained estimate of the noise.

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